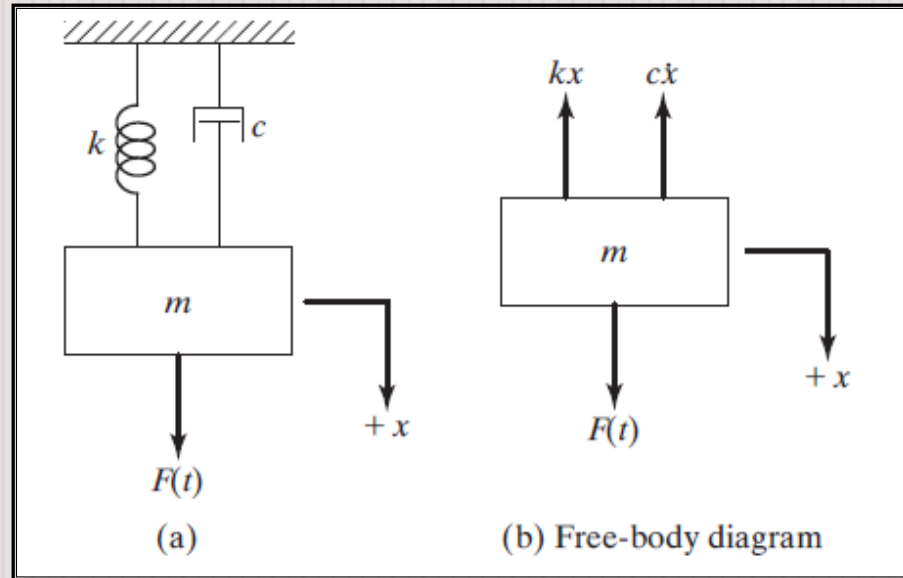


Mechanical Vibrations

Forced Vibration of a Single Degree of Freedom System

Harmonically Excited Vibration

- Physical system



$$m \ddot{x} + c \dot{x} + kx = F(t) \quad \text{Equation of motion}$$

$F(t)$ is a harmonic force. It may takes one of the following forms:

$$F(t) = F_o \sin \omega t$$

$$F(t) = F_o \cos \omega t$$

$$F(t) = F_o e^{j\omega t}$$

F_o is the amplitude of force

ω is the frequency of force

Harmonically Excited Vibration

Consider the form:

$$m \ddot{x} + c \dot{x} + kx = F_o e^{j\omega t} \quad \text{Equation of motion (1)}$$

This 2nd order differential equation is non-homogeneous. Its solution has two parts

1. Complementary function (Transient response): It is the solution for the homogeneous equation (described in previous chapter). for $\zeta < 1$

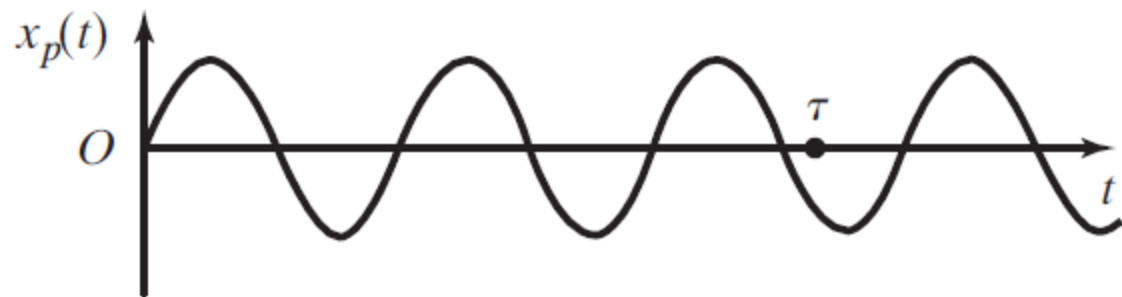
The solution was written as:

$$x_h(t) = e^{-\zeta\omega_n t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

2. Particular integral (steady state response): It is the solution for the non homogeneous equation. Because the force (excitation) is harmonic with frequency ω we can expect that the response (solution) is harmonic with the same frequency ω . The solution can be written as

$$x_p(t) = \bar{X} e^{j\omega t}, \quad \bar{X} \text{ is called the 'steady state response amplitude'}$$

Harmonically Excited Vibration



Harmonically Excited Vibration

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{j\omega t}$$

$$x(t) = \bar{X} e^{j\omega t}$$

$$-m\omega^2 \bar{X} e^{j\omega t} + jc\omega \bar{X} e^{j\omega t} + k\bar{X} e^{j\omega t} = F_0 e^{j\omega t}$$

$$(k - m\omega^2 + jc\omega) \bar{X} e^{j\omega t} = F_0 e^{j\omega t}$$

$$(k - m\omega^2 + jc\omega) \bar{X} = F_0$$

Harmonically Excited Vibration

Substitute the particular integral into equation (1) and solve for $\bar{X}$:

$$\therefore \bar{X} = \frac{F_o}{(k - \omega^2 m) + jc\omega}$$

By writing $(k - \omega^2 m) + jc\omega$ as

$$\sqrt{(k - \omega^2 m)^2 + (c\omega)^2} e^{j\phi}, \text{ where } \phi = \tan^{-1} \frac{c\omega}{k - \omega^2 m} \text{ is the "phase$$

angle" between Force and Response.}

$$\therefore \bar{X} = |\bar{X}| e^{-j\phi} \quad \text{where } |\bar{X}| = \frac{F_o}{\sqrt{(k - \omega^2 m)^2 + (c\omega)^2}}$$

There for the particular integral can now be written as

$$x_p = |\bar{X}| e^{j(\omega t - \phi)}, \quad |\bar{X}| \text{ denotes the "magnitude" of the steady state response amplitude.}$$

Harmonically Excited Vibration

$|\bar{X}|$ and φ may be written as:

$$|\bar{X}| = \frac{\left(\frac{F_o}{k}\right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \quad \text{where } r = \frac{\omega}{\omega_n} \text{ is the frequency ratio.}$$

$$\varphi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

The total solution (response) for the equation of motion (1) is:

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = e^{-\zeta\omega_n t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t) + |\bar{X}| e^{j(\omega t - \varphi)}$$

Note that C_1 and C_2 are constants to be determined from knowing the initial conditions of the motion.

Harmonically Excited Vibration

NOTES:

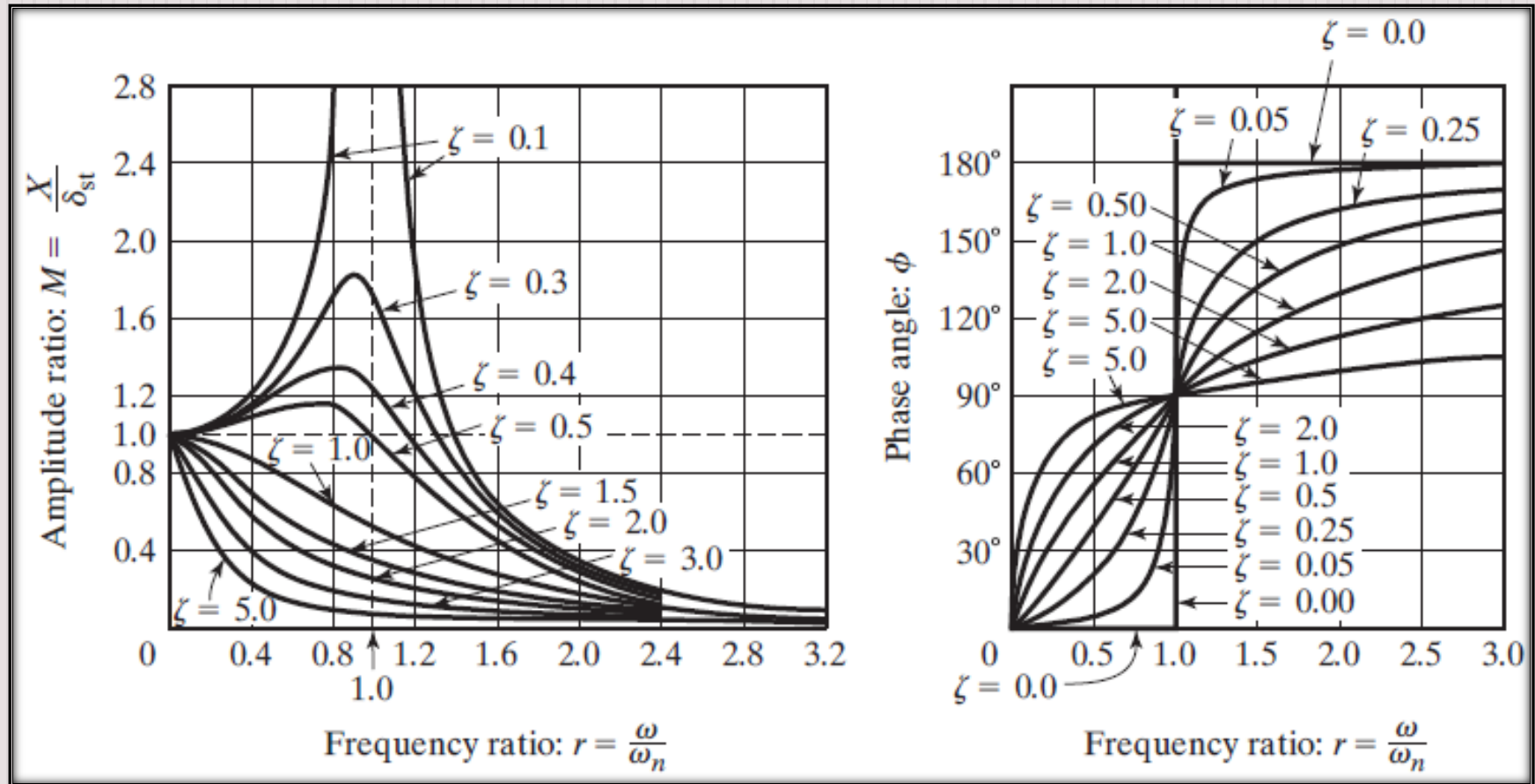
1. For the force form $F(t) = F_o \sin \omega t \Rightarrow x_p(t) = |\bar{X}| \sin(\omega t - \varphi)$
2. For the force form $F(t) = F_o \cos \omega t \Rightarrow x_p(t) = |\bar{X}| \cos(\omega t - \varphi)$
3. Transient response ($x_h(t)$) represents a motion that decays with time and can be neglected after a certain time.
4. Steady state response ($x_p(t)$) is a harmonic motion with constant amplitude and frequency ω .
5. The term $\frac{F_o}{k}$ is usually called the "static deflection (δ_{st})". The ratio

$\frac{|\bar{X}|}{\delta_{st}}$ is usually called the "Magnification Factor (M)".

$$M = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Damped Forced Vibration System

- Graphical representation for Magnification factor M and ϕ .



Damped Forced Vibration System

- Notes on the graphical representation of X.
 - ❑ For $\zeta = 0$, the system is reduced and becomes un-damped.
 - ❑ for any amount of $\zeta > 0$, the amplitude of vibration decreases (i.e. reduction in the magnification factor M). This is correct for any value of r .
 - ❑ For the case of $r = 0$, the magnification factor equals 1.
 - ❑ The amplitude of the forced vibration approaches zero when the frequency ratio 'r' approaches the infinity (i.e. $M \rightarrow 0$ when $r \rightarrow \infty$)

Damped Forced Vibration System

- Notes on the graphical representation for ϕ .
 - ❑ For $\zeta = 0$, the phase angle is zero for $0 < r < 1$ and 180° for $r > 1$.
 - ❑ For any amount of $\zeta > 0$ and $0 < r < 1$, $0^\circ < \phi < 90^\circ$.
 - ❑ For $\zeta > 0$ and $r > 1$, $90^\circ < \phi < 180^\circ$.
 - ❑ For $\zeta > 0$ and $r = 1$, $\phi = 90^\circ$.
 - ❑ For $\zeta > 0$ and $r \gg 1$, ϕ approaches 180° .

Harmonically Excited Vibration

Variation of displacement with frequency :

We have :

$$|\bar{X}| = \frac{\frac{F_o}{k}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \text{ is the s.s response amplitude}$$

1. At low frequency ($r \rightarrow 0$) then $|\bar{X}| \approx \frac{F_o}{k}$ which means the displacement is ***Stiffness control***.
2. At high frequency ($r \gg 1$) then $|\bar{X}| \approx \frac{F_o}{m\omega^2}$ which means the displacement is ***Mass control***.
3. At resonance ($r = 1$) then $|\bar{X}| \approx \frac{F_o}{2\zeta k}$ which means the displacement is ***Damping control***. (for small ζ)

Harmonically Excited Vibration

Frequency of Maximum amplitude :

The s.s response amplitude can be written as :

$$|\bar{X}| = \frac{F_o}{k} \left((1 - r^2)^2 + (2\zeta r)^2 \right)^{1/2}$$

$$|\bar{X}| \text{ is maximum when } \frac{d}{dr} (|\bar{X}|) = 0$$

This condition gives :

$$|\bar{X}| \text{ is maximum when } r = \frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2} \text{ which is known as a}$$

$$\textbf{Resonance Frequency, i.e} \quad \omega_{\text{res}} = \omega_n \sqrt{1 - 2\zeta^2}$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$m\ddot{x} + kx = F_0 \cos \omega t$$

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$

$$x_p(t) = X \cos \omega t$$



$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

Using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \quad C_2 = \frac{\dot{x}_0}{\omega_n}$$

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

$$\xrightarrow{\begin{matrix} x_0=0 \\ \dot{x}_0=0 \end{matrix}}$$

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$$

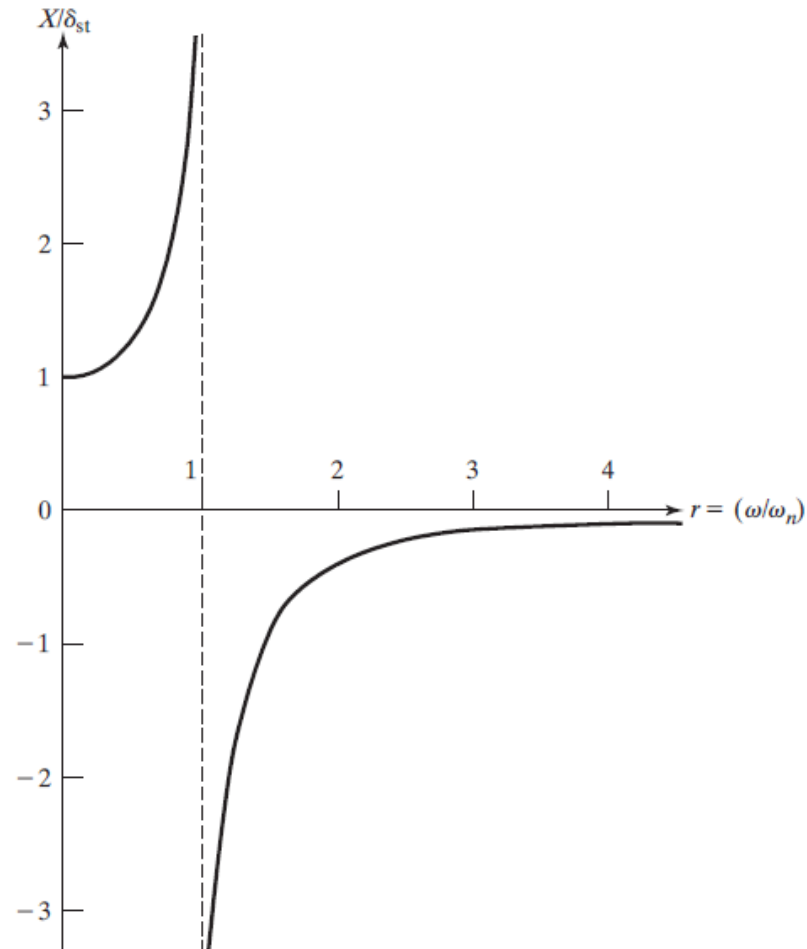


$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

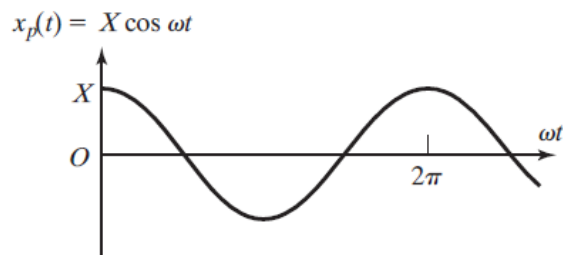
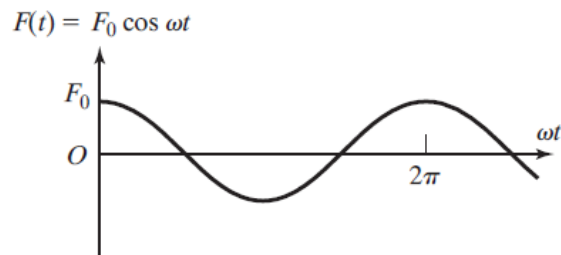
$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$



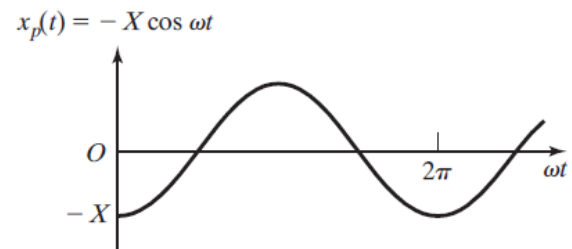
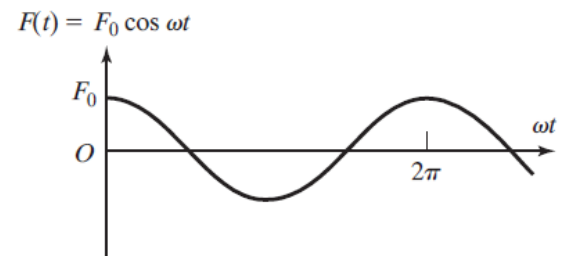
Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$x_p(t) = \frac{F_0}{k - m\omega^2} \cos \omega t$$



$$0 < \omega / \omega_n < 1.$$



$$\omega / \omega_n > 1.$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right]$$

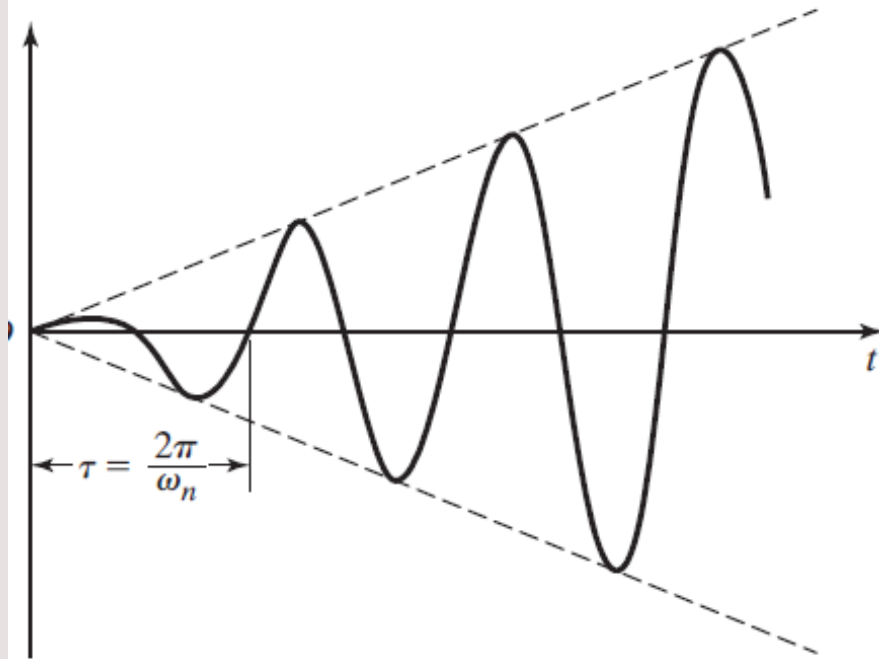
When $\omega/\omega_n = 1$

$$\lim_{\omega \rightarrow \omega_n} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \rightarrow \omega_n} \left[\frac{\frac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\frac{d}{d\omega} \left(1 - \frac{\omega^2}{\omega_n^2} \right)} \right] = \lim_{\omega \rightarrow \omega_n} \left[\frac{t \sin \omega t}{2 \frac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t$$



Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

Beating Phenomenon

If the forcing frequency is close to, but not exactly equal to, the natural frequency

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

$$x_0 = \dot{x}_0 = 0$$

$$x(t) = \frac{(F_0/m)}{\omega_n^2 - \omega^2} (\cos \omega t - \cos \omega_n t) = \frac{(F_0/m)}{\omega_n^2 - \omega^2} \left[2 \sin \frac{\omega + \omega_n}{2} t \cdot \sin \frac{\omega_n - \omega}{2} t \right]$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

Beating Phenomenon

Let the forcing frequency ω be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon$$

where ε is a small positive quantity. Then $\omega_n \approx \omega$ and

$$\omega + \omega_n \simeq 2\omega$$

Multiplication of Eqs. (3.19) and (3.20) gives

$$\omega_n^2 - \omega^2 = 4\varepsilon\omega$$

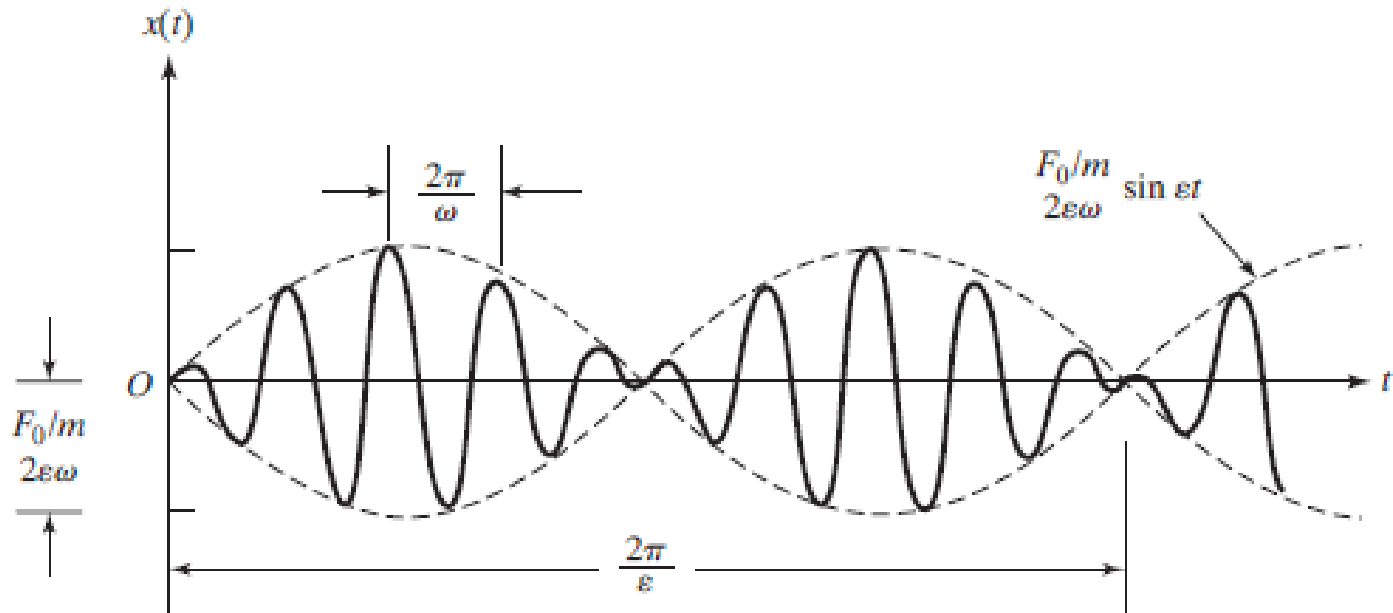
The use of Eqs. (3.19) to (3.21) in Eq. (3.18) gives

$$x(t) = \left(\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t \right) \sin \omega t$$

Harmonically Excited Vibration

Response of an Undamped System Under Harmonic Force

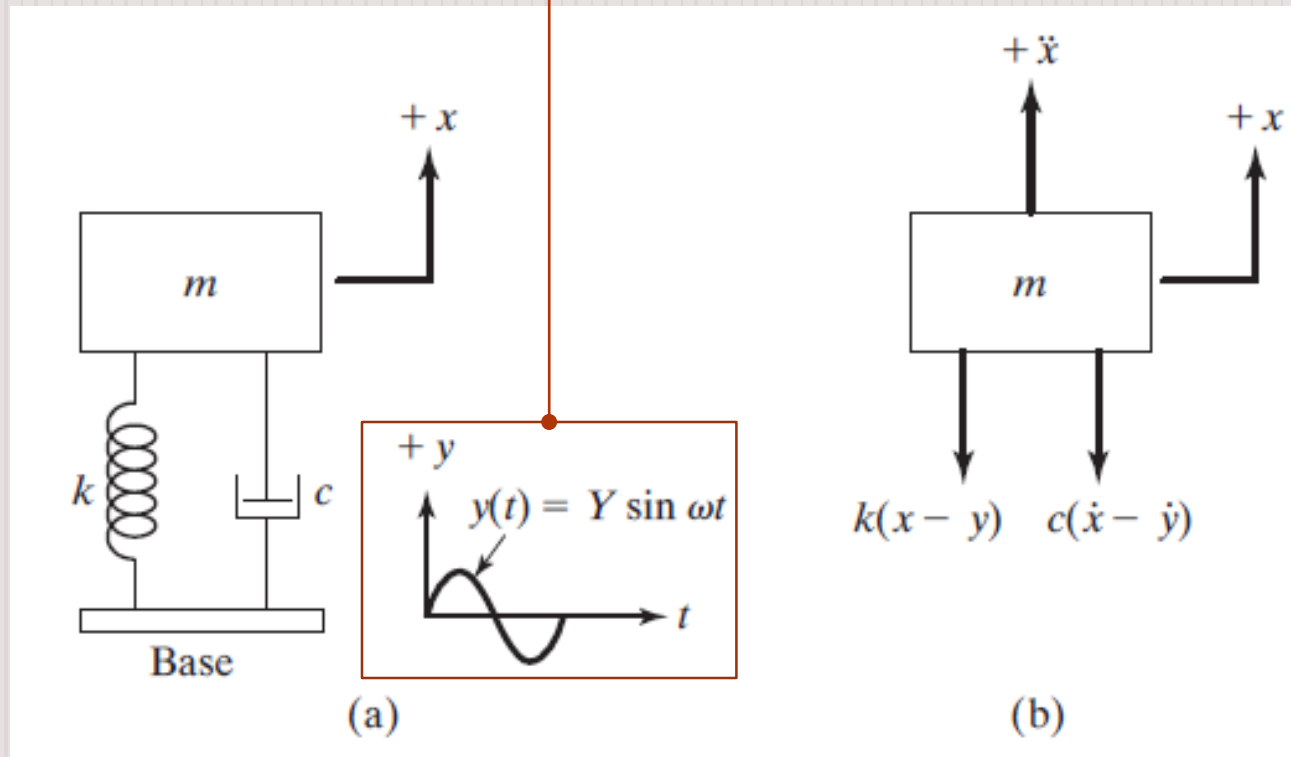
Beating Phenomenon



Transmissibility of displacement (support motion)

Physical system:

The forcing function for the base excitation



Mathematical model:
$$m \ddot{x} + c \left(\dot{x} - \dot{y} \right) + k(x - y) = 0$$

Transmissibility of displacement (support motion)

Substitute the forcing function into the math. Model:

$$m \ddot{x} + c \dot{x} + kx = kY \sin(\omega t) + c\omega Y \cos(\omega t) = A \sin(\omega t - \alpha)$$

Where :

$$A = Y \sqrt{k^2 + (c\omega)^2}, \quad \alpha = \tan^{-1}\left(-\frac{c\omega}{k}\right)$$

Assume $x_p(t) = X \sin(\omega t - \alpha - \phi_1)$ (Harmonic motion)

$$x_p(t) = \frac{Y \sqrt{k^2 + (c\omega)^2}}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \sin(\omega t - \phi_1 - \alpha)$$

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

Transmissibility of displacement (support motion)

$$x_p(t) = X \sin(\omega t - \phi)$$

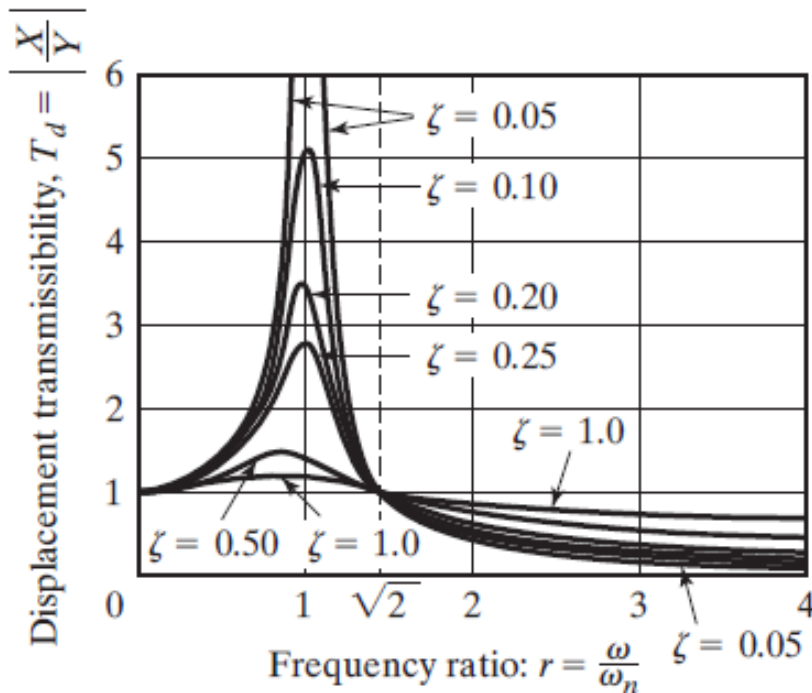
$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2} \right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

$$\phi = \tan^{-1} \left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2} \right] = \tan^{-1} \left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2} \right]$$

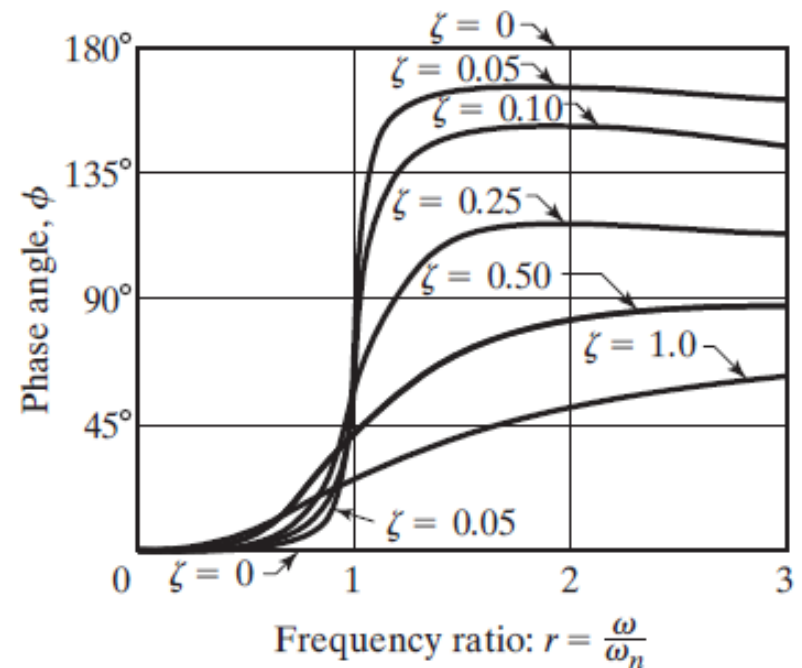
The ratio of the amplitude of the response $x_p(t)$ to that of the base motion $y(t)$, $\frac{X}{Y}$, is called the *displacement transmissibility*. $\frac{X}{Y} \equiv T_d$

Transmissibility of displacement (support motion)

Graphical representation of Force or Displacement Transmissibility ((TR) and the Phase angle (ϕ)



(a)



(b)

Transmissibility of displacement (support motion)

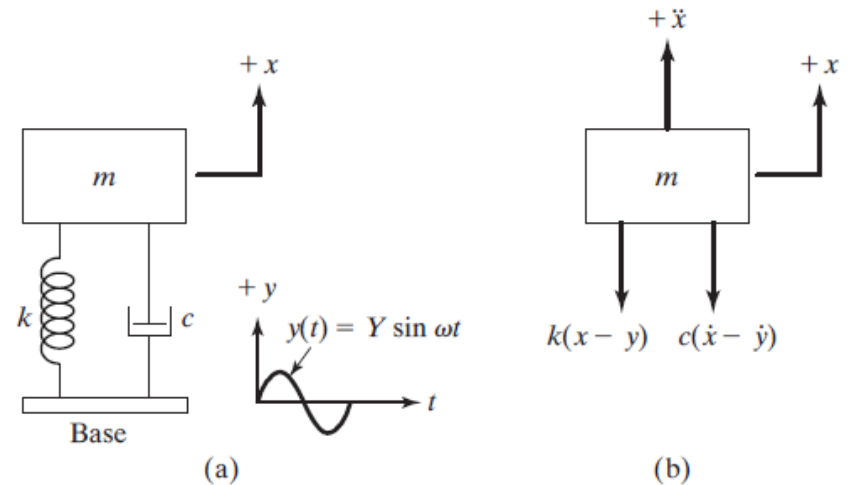
The following aspects of *displacement transmissibility*, $T_d = \frac{X}{Y}$

1. The value of T_d is unity at $r = 0$ and close to unity for small values of r .
2. For an undamped system ($\zeta = 0$), $T_d \rightarrow \infty$ at resonance ($r = 1$).
3. The value of T_d is less than unity ($T_d < 1$) for values of $r > \sqrt{2}$ (for any amount of damping ζ).
4. The value of T_d is unity for all values of ζ at $r = \sqrt{2}$.
5. For $r < \sqrt{2}$, smaller damping ratios lead to larger values of T_d . On the other hand, for $r > \sqrt{2}$, smaller values of damping ratio lead to smaller values of T_d .
6. The displacement transmissibility, T_d , attains a maximum for $0 < \zeta < 1$ at the frequency ratio $r = r_m < 1$ given by (see Problem 3.60):

$$r_m = \frac{1}{2\zeta} \left[\sqrt{1 + 8\zeta^2} - 1 \right]^{1/2}$$

Transmissibility of displacement (support motion)

Force Transmitted



$$F = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x}$$

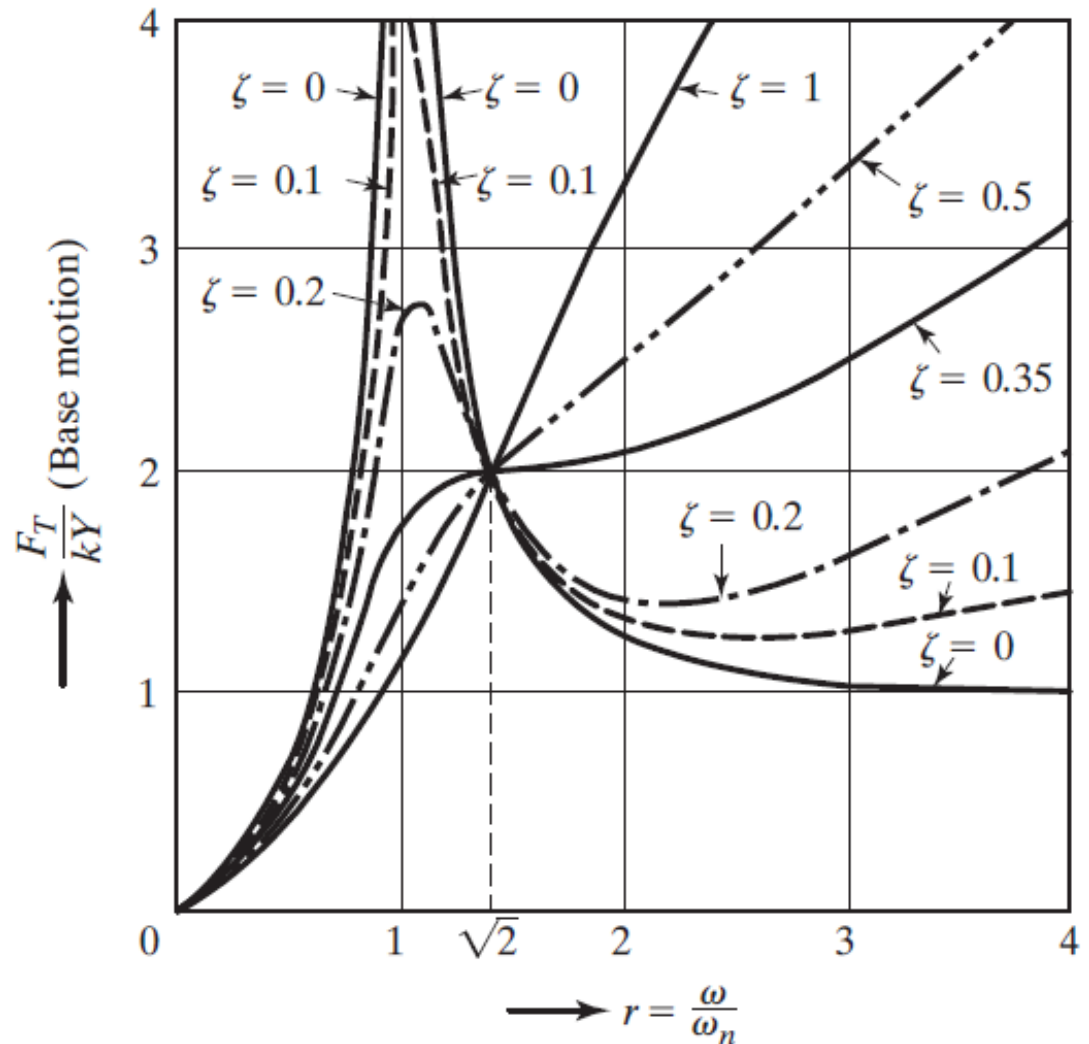
$$x_p(t) = X \sin(\omega t - \phi)$$

$$F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi)$$

Transmissibility of displacement (support motion)

$$X = Y \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$



Transmissibility of displacement (support motion)

Relative Motion

$$m \ddot{x} + c \left(\dot{x} - \dot{y} \right) + k(x - y) = 0$$

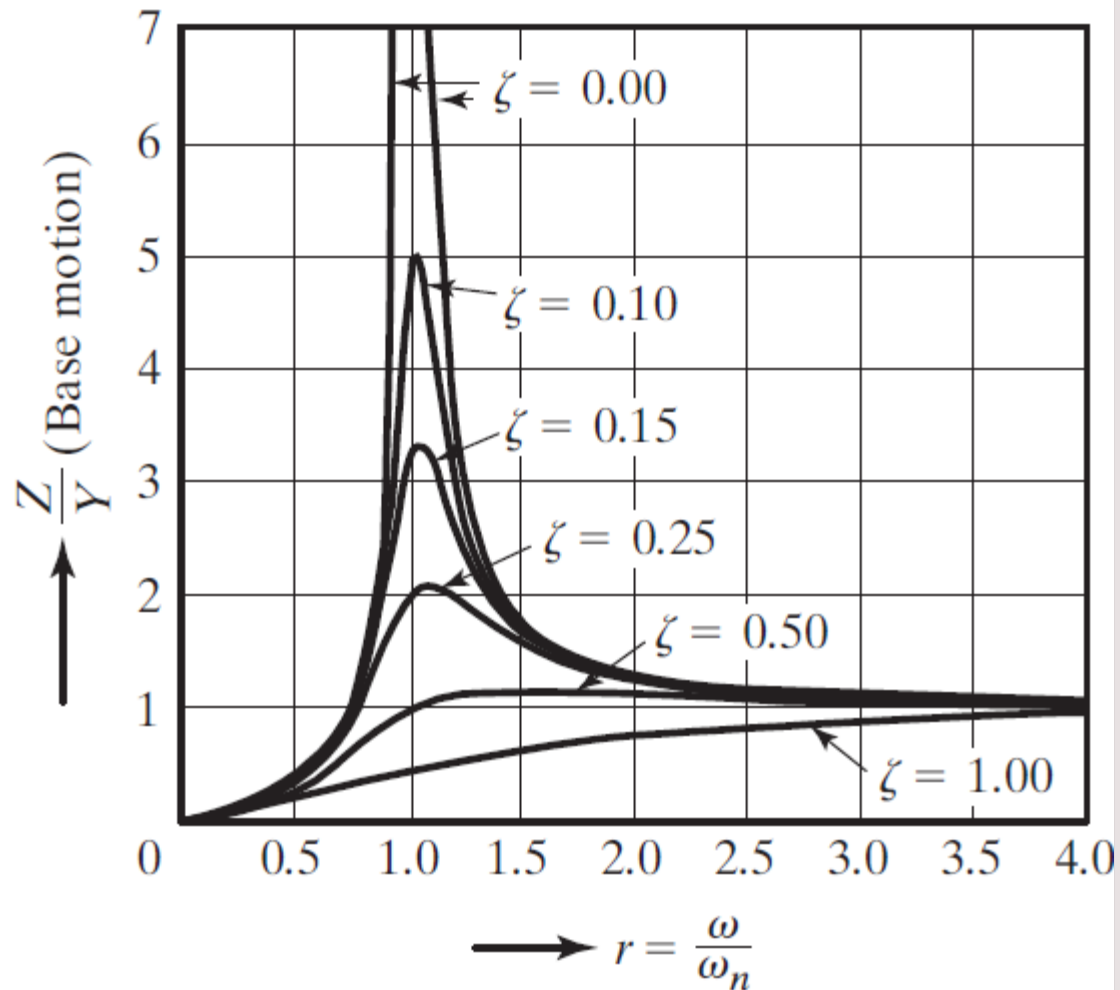
$$z = x - y \longrightarrow m \ddot{z} + c \dot{z} + kz = -m \ddot{y} = m \omega^2 Y \sin \omega t$$

$$z(t) = \frac{m \omega^2 Y \sin(\omega t - \phi_1)}{[(k - m \omega^2)^2 + (c \omega)^2]^{1/2}} = Z \sin(\omega t - \phi_1)$$

$$Z = \frac{m \omega^2 Y}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}} = Y \frac{r^2}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}}$$

$$\phi_1 = \tan^{-1} \left(\frac{c \omega}{k - m \omega^2} \right) = \tan^{-1} \left(\frac{2 \zeta r}{1 - r^2} \right)$$

Transmissibility of displacement (support motion)

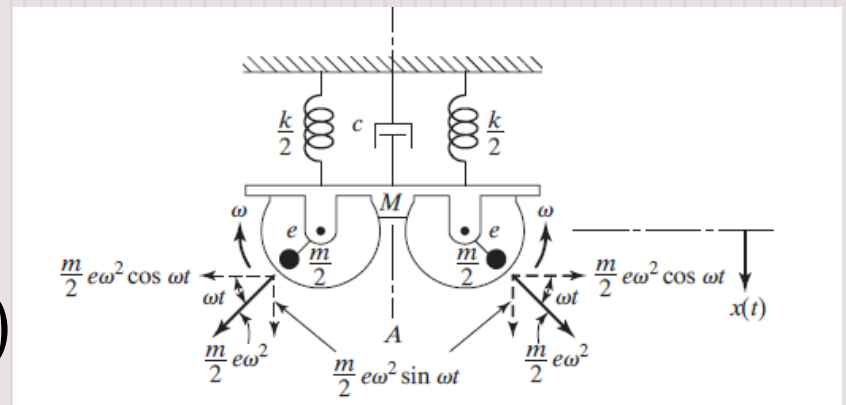
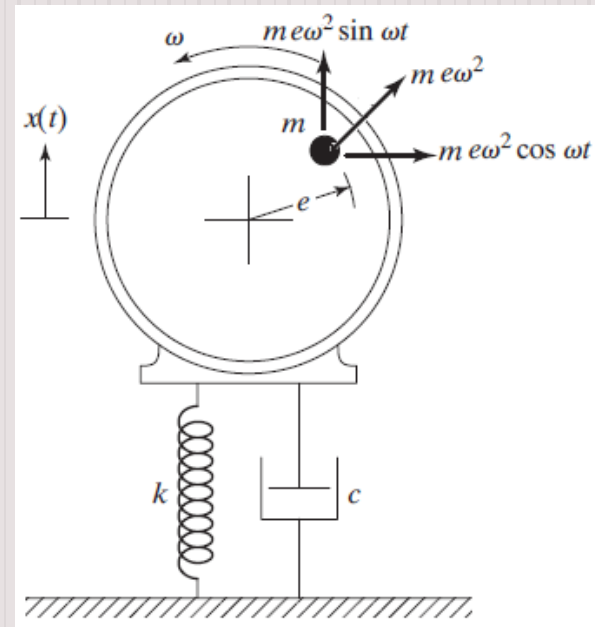


Forced Vibration due to Rotating Unbalance

Unbalance in rotating machines is a common source of vibration excitation. If M_t is the total mass of the system, m is the eccentric mass and ω is the speed of rotation, the centrifugal force due to unbalanced mass is $me\omega^2$ where e is the eccentricity.

The vertical component ($me\omega^2 \sin(\omega t)$) is the effective one because it is in the direction of motion of the system. The equation of motion is:

$$M_t \ddot{x} + c \dot{x} + kx = me\omega^2 \sin(\omega t)$$



Forced Vibration due to Rotating Unbalance

$$x(t) = X \sin(\omega t - \varphi)$$

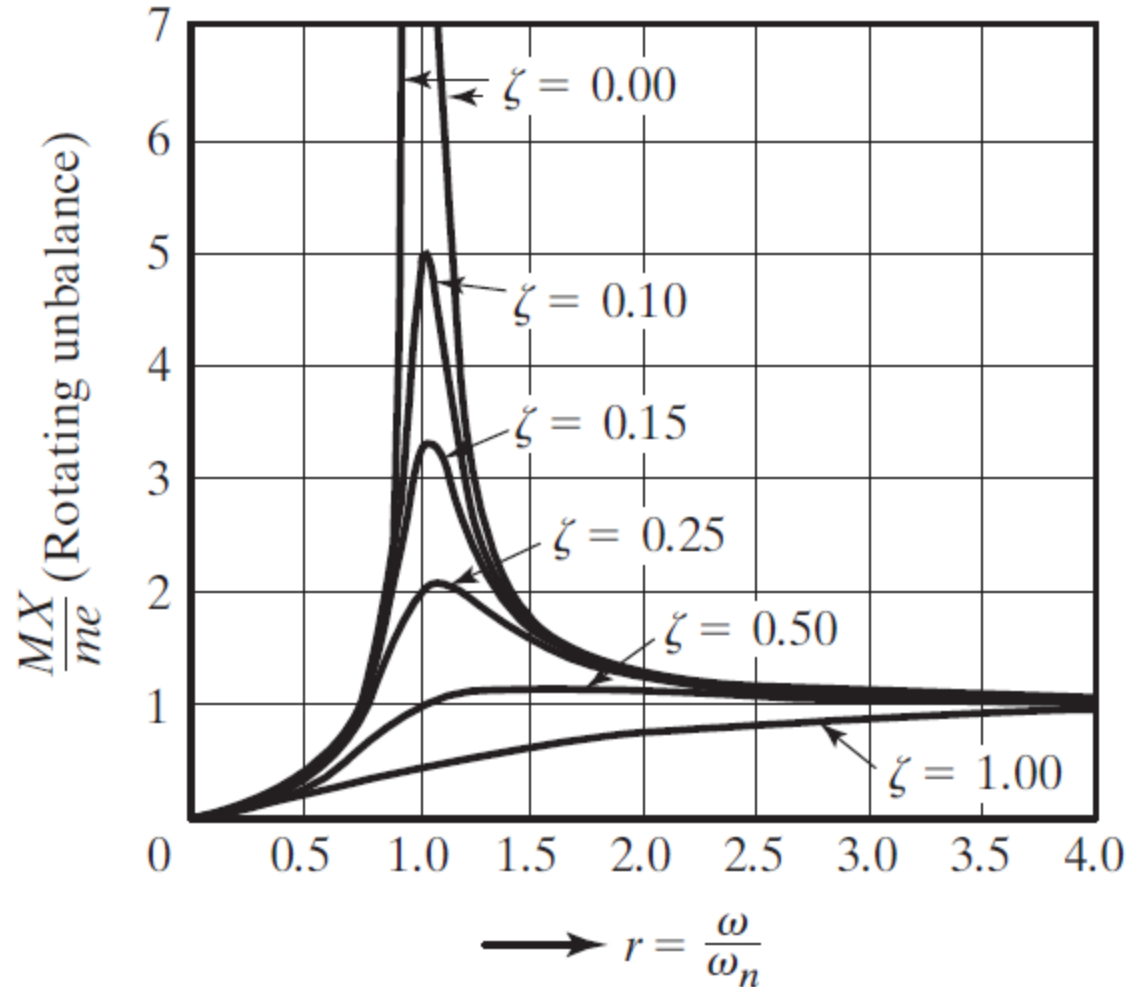
$$X = \frac{\left(\frac{me\omega^2}{k} \right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \quad \text{or}$$

$$X = \frac{\left(\frac{me}{M_t} \right) r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \rightarrow \frac{M_t X}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

and

$$\varphi = \tan^{-1} \left(\frac{2\zeta r}{1-r^2} \right)$$

Forced Vibration due to Rotating Unbalance



Forced Vibration due to Rotating Unbalance

1. All the curves begin at zero amplitude. The amplitude near resonance ($\omega = \omega_n$) is markedly affected by damping. Thus if the machine is to be run near resonance, damping should be introduced purposefully to avoid dangerous amplitudes.
2. At very high speeds (ω large), MX/me is almost unity, and the effect of damping is negligible.
3. For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum of $\frac{MX}{me}$ occurs when

$$\frac{d}{dr} \left(\frac{MX}{me} \right) = 0$$

The solution of Eq. (3.82) gives

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} > 1$$

with the corresponding maximum value of $\frac{MX}{me}$ given by

$$\left(\frac{MX}{me} \right)_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Thus the peaks occur to the right of the resonance value of $r = 1$.

4. For $\zeta > \frac{1}{\sqrt{2}}$, $\left[\frac{MX}{me} \right]$ does not attain a maximum. Its value grows from 0 at $r = 0$ to 1 at $r \rightarrow \infty$.

Transmissibility of Force

If the deflection of foundation is negligible, then the force transmitted to the foundation is

$$F_{tr} e^{j\omega t} = kx + c\dot{x}$$

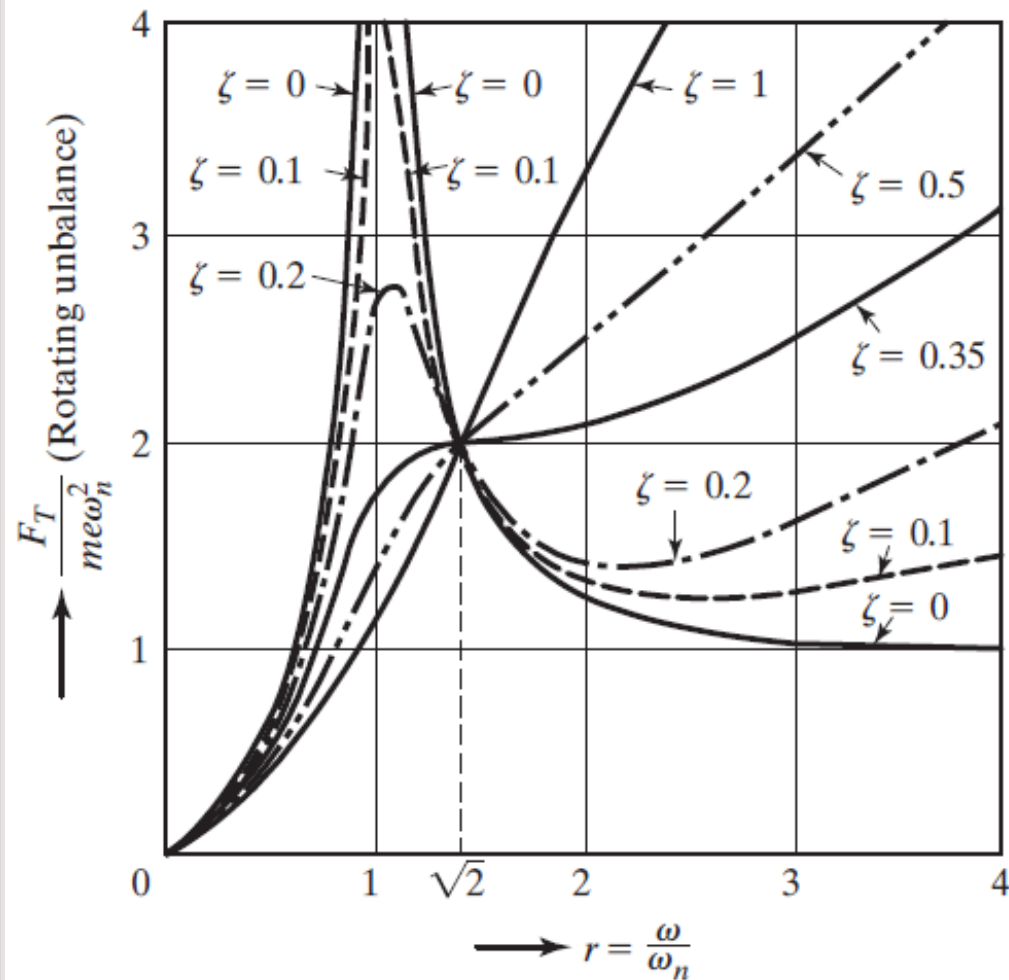
Assume harmonic motion i.e, $x = X e^{j\omega t}$

$$F_{tr} = \frac{(k + j\omega c) m e \omega^2}{((k - \omega^2 M_t) + j\omega c)}$$

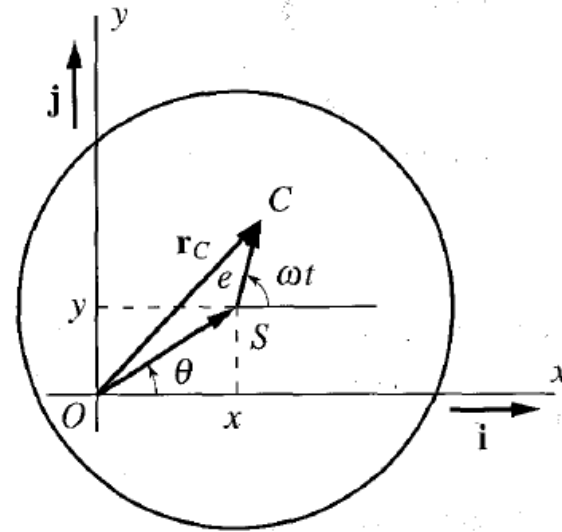
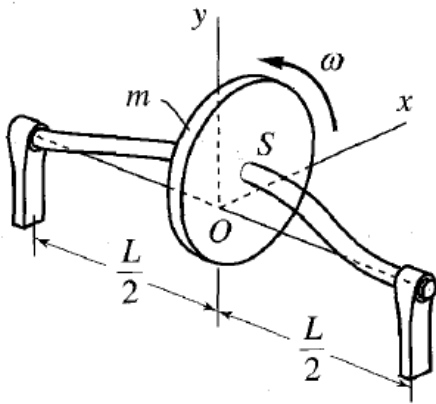
$$|F_{tr}| = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} m e \omega^2$$

$$\rightarrow TR_F = \frac{|F_{tr}|}{m e \omega_n^2} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} r^2$$

Transmissibility of Force



Whirling of Rotating Shafts



$$\mathbf{r}_C = (x + e \cos \omega t) \mathbf{i} + (y + e \sin \omega t) \mathbf{j}$$

$$\mathbf{a}_C = (\ddot{x} - e\omega^2 \cos \omega t) \mathbf{i} + (\ddot{y} - e\omega^2 \sin \omega t) \mathbf{j}$$

Whirling of Rotating Shafts

To derive the equations of motion, we assume that the only forces acting on the disk are restoring forces due to the elasticity of the shaft and resisting forces due to viscous damping, such as caused by air friction. The elastic effects are represented by equivalent spring constants k_x and k_y associated with the deformation of the shaft in the x and y directions, respectively. Moreover, we assume that the coefficient of viscous damping is the same in both directions and equal to c . The elastically restoring forces and the viscous damping forces are acting at point S .

$$-k_x x - c\dot{x} = m(\ddot{x} - e\omega^2 \cos \omega t)$$

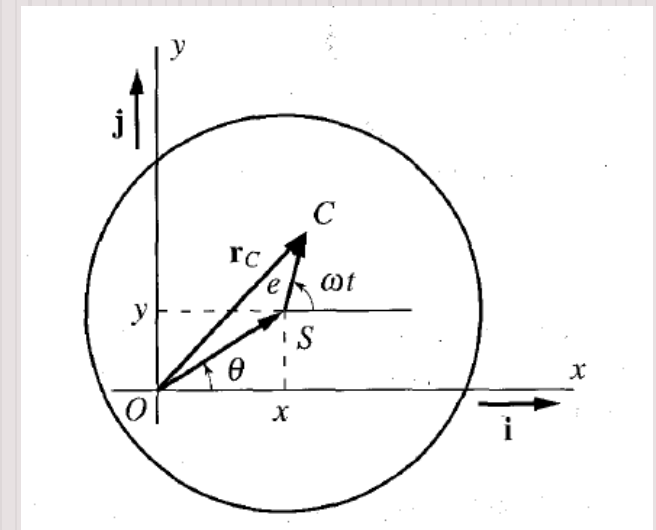
$$-k_y y - c\dot{y} = m(\ddot{y} - e\omega^2 \sin \omega t)$$

$$\ddot{x} + 2\zeta_x \omega_{nx} \dot{x} + \omega_{nx}^2 x = e\omega^2 \cos \omega t$$

$$\ddot{y} + 2\zeta_y \omega_{ny} \dot{y} + \omega_{ny}^2 y = e\omega^2 \sin \omega t$$

$$\zeta_x = \frac{c}{2m\omega_{nx}}, \quad \omega_{nx} = \sqrt{\frac{k_x}{m}}$$

$$\zeta_y = \frac{c}{2m\omega_{ny}}, \quad \omega_{ny} = \sqrt{\frac{k_y}{m}}$$



Whirling of Rotating Shafts

$$\ddot{x} + 2\zeta_x\omega_{nx}\dot{x} + \omega_{nx}^2x = e\omega^2 \cos\omega t$$

$$x(t) = |X(\omega)| \cos(\omega t - \phi_x)$$

$$|G_x(i\omega)| = \frac{1}{\{[1 - (\omega/\omega_{nx})^2]^2 + (2\zeta_x\omega/\omega_{nx})^2\}^{1/2}}$$

$$\phi_x = \tan^{-1} \frac{2\zeta_x\omega/\omega_{nx}}{1 - (\omega/\omega_{nx})^2}$$

$$\ddot{y} + 2\zeta_y\omega_{ny}\dot{y} + \omega_{ny}^2y = e\omega^2 \sin\omega t$$

$$y(t) = |Y(\omega)| \sin(\omega t - \phi_y)$$

$$|G_y(i\omega)| = \frac{1}{\{[1 - (\omega/\omega_{ny})^2]^2 + (2\zeta_y\omega/\omega_{ny})^2\}^{1/2}}$$

$$\phi_y = \tan^{-1} \frac{2\zeta_y\omega/\omega_{ny}}{1 - (\omega/\omega_{ny})^2}$$

Whirling of Rotating Shafts

We consider first the most common case, namely, that of a shaft of circular cross section, so that the stiffness is the same in both directions, $k_x = k_y = k$.

$$\omega_{nx} = \omega_{ny} = \omega = \sqrt{\frac{k}{m}}, \quad \zeta_x = \zeta_y = \zeta = \frac{c}{2m\omega_n}$$

$$|G_x(i\omega)| = |G_y(i\omega)| = |G(i\omega)| = \frac{1}{\{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2\}^{1/2}}$$
$$\phi_x = \phi_y = \phi = \tan^{-1} \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}$$

$$|X(\omega)| = |Y(\omega)| = e \left(\frac{\omega}{\omega_n} \right)^2 |G(i\omega)|$$



$$x(t) = e \left(\frac{\omega}{\omega_n} \right)^2 |G(i\omega)| \cos(\omega t - \phi)$$

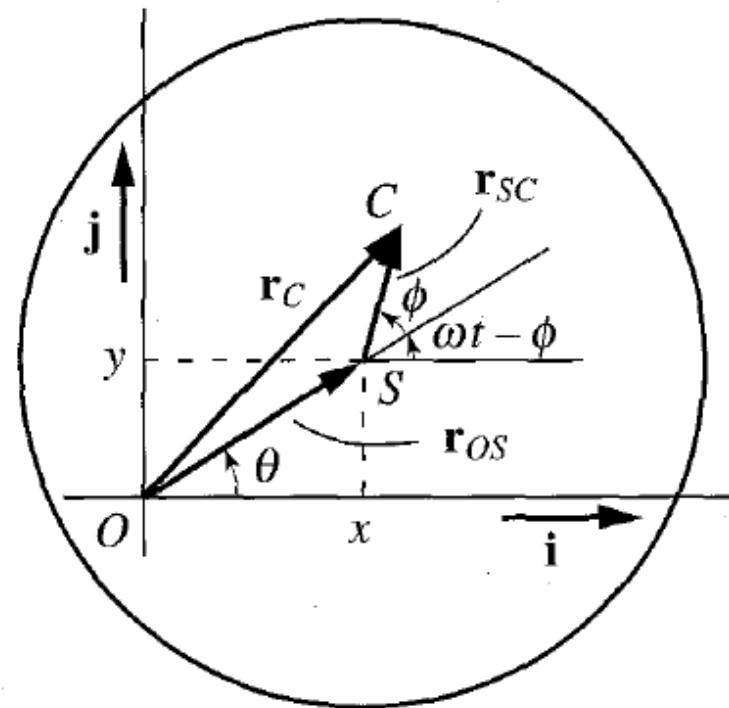
$$y(t) = e \left(\frac{\omega}{\omega_n} \right)^2 |G(i\omega)| \sin(\omega t - \phi)$$

Whirling of Rotating Shafts

$$\tan \theta = \frac{y}{x} = \tan(\omega t - \phi)$$

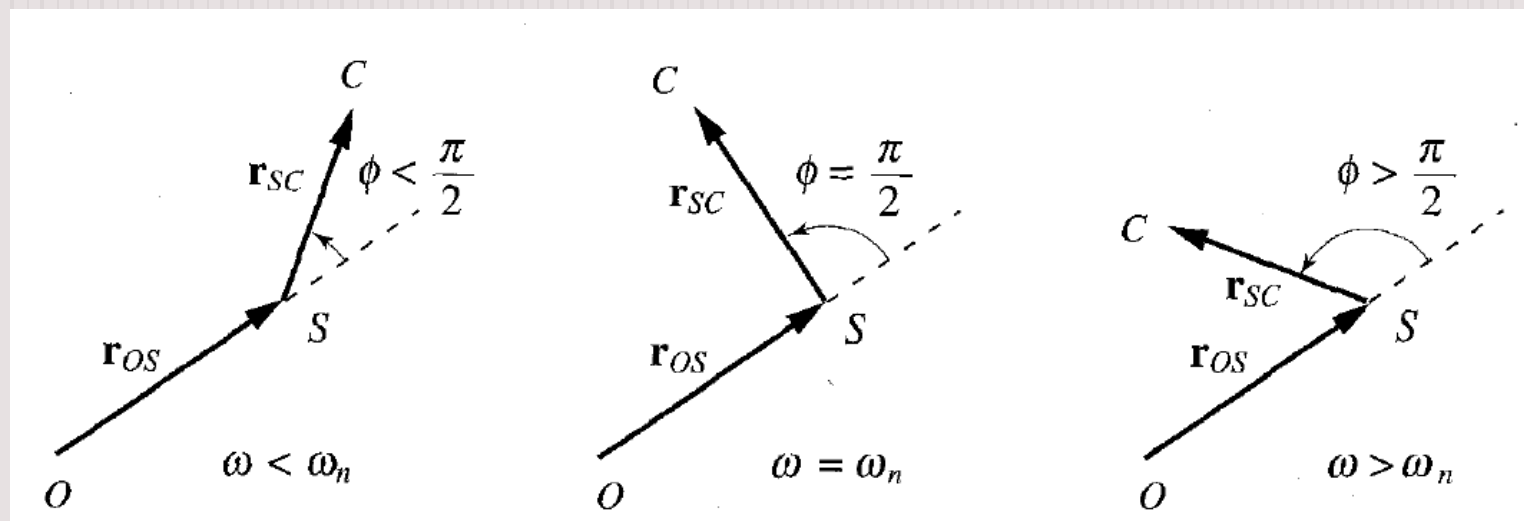
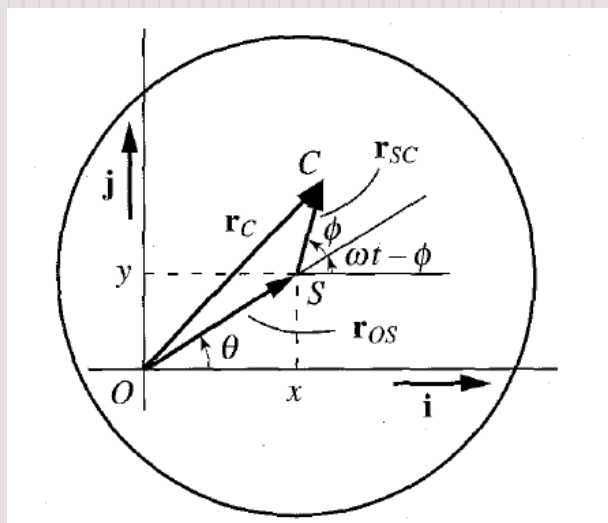
$$\theta = \omega t - \phi$$

$$\dot{\theta} = \omega$$



$$r_{OS} = \sqrt{x^2 + y^2} = e \left(\frac{\omega}{\omega_n} \right)^2 |G(i\omega)| = \text{constant}$$

Whirling of Rotating Shafts



Whirling of Rotating Shafts

Next, we return to the case in which the two stiffnesses are different and consider the undamped case, $c = 0$.

$$x(t) = |X(\omega)| \cos(\omega t - \phi_x)$$

$$y(t) = |Y(\omega)| \sin(\omega t - \phi_y)$$



$$x(t) = X(\omega) \cos \omega t$$

$$y(t) = Y(\omega) \sin \omega t$$

$$X(\omega) = \frac{e(\omega/\omega_{nx})^2}{1 - (\omega/\omega_{nx})^2}$$

$$Y(\omega) = \frac{e(\omega/\omega_{ny})^2}{1 - (\omega/\omega_{ny})^2}$$



$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

$$\tan \theta = \frac{y}{x} = \frac{Y}{X} \tan \omega t$$

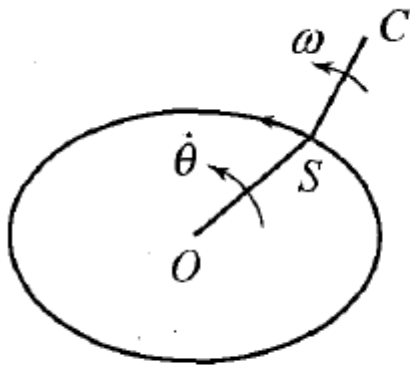
Whirling of Rotating Shafts

$$\tan \theta = \frac{y}{x} = \frac{Y}{X} \tan \omega t$$

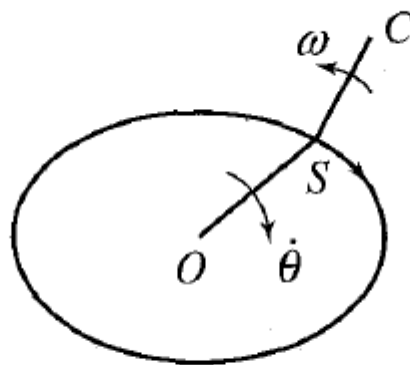


$$\dot{\theta} = \frac{XY}{X^2 \cos^2 \omega t + Y^2 \sin^2 \omega t} \omega$$

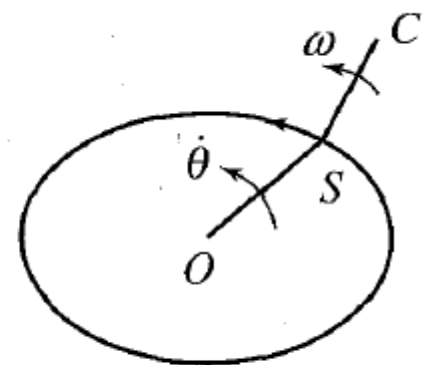
1. $\omega < \omega_{nx}$ and $\omega < \omega_{ny}$. In this case, we conclude from Eqs. (3.76) that $XY > 0$, so that point S moves on the ellipse in the same sense as the rotation ω .
2. $\omega_{nx} < \omega < \omega_{ny}$ or $\omega_{ny} < \omega < \omega_{nx}$. In either of these two cases $XY < 0$, so that S moves in the opposite sense from ω .
3. $\omega > \omega_{nx}$ and $\omega > \omega_{ny}$. In this case $XY > 0$, so that S moves in the same sense as ω .



$\omega < \omega_{nx}$ and $\omega < \omega_{ny}$



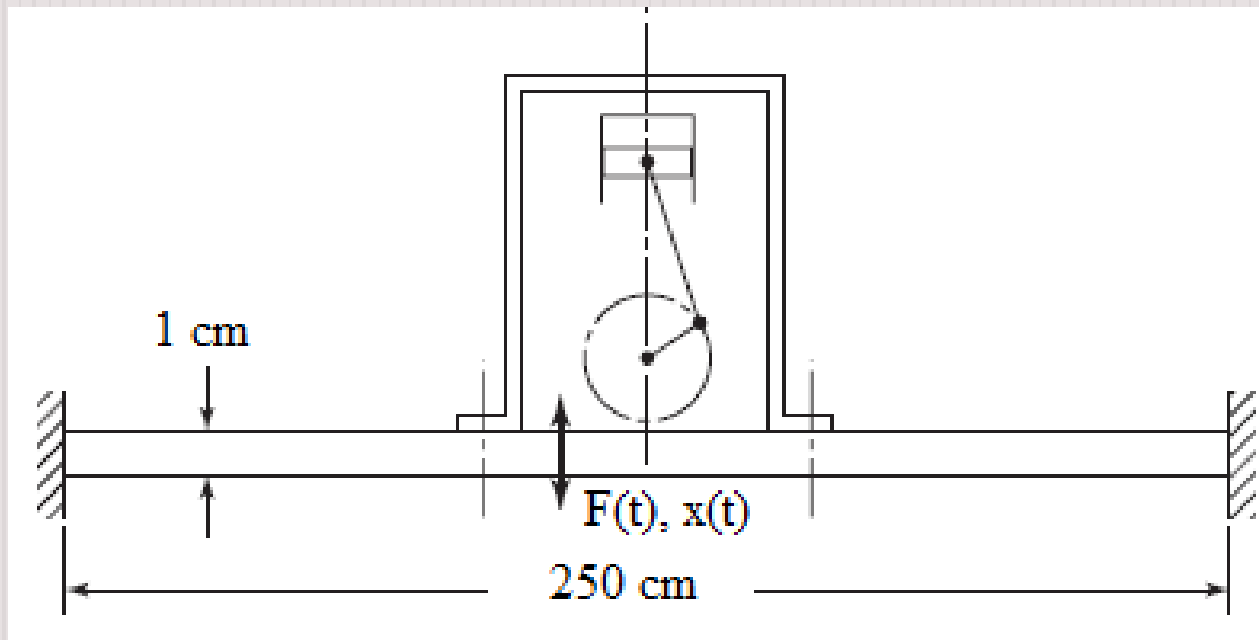
$\omega_{nx} < \omega < \omega_{ny}$ or $\omega_{ny} < \omega < \omega_{nx}$



$\omega > \omega_{nx}$ and $\omega > \omega_{ny}$

- **Example 3.1: Plate Supporting a Pump:**

A reciprocating pump, weighing 68 kg, is mounted at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm. clamped along two edges as shown in Fig. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 220 \cos (62.832t)$ N. if $E=200$ Gpa, *Find the amplitude of vibration of the plate.*



- **Example 3.1: solution**

- The plate can be modeled as fixed – fixed beam has the following stiffness:

$$k = \frac{192EI}{l^3}$$

$$\text{But } I = \frac{1}{12}bh^3 = \frac{1}{12}(50 \times 10^{-2})(1 \times 10^{-2})^3 = 41.667 \times 10^{-9} \text{ m}^4$$

$$\text{So, } k = \frac{192(200 \times 10^9)(41.667 \times 10^{-9})}{(250 \times 10^{-2})^3} = 102,400.82 \text{ N / m}$$

- The maximum amplitude (X) is found as:

$$X = \frac{F_o}{k - m\omega^2} = \frac{220}{102,400.82 - 68(62.832)} = -1.32487 \text{ mm}$$

Example 3.2:

Find the total response of a single-degree-of-freedom system with $m = 10 \text{ kg}$, $c = 20 \text{ N-s/m}$, $k = 4000 \text{ N/m}$, $x_o = 0.01 \text{ m}$ and $\dot{x}_o = 0$ when an external force $F(t) = F_o \cos(\omega t)$ acts on the system with $F_o = 100 \text{ N}$ and $\omega = 10 \text{ rad/sec}$.

Solution

a. From the given data

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{20}{2(10)(20)} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n$$

$$\omega_d = \sqrt{1 - 0.05^2} (20)$$

$$\omega_d = 19.975 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

Example 3.2:

Solution

$$\delta_{st} = \frac{F_o}{k} = \frac{100}{4000} = 0.025m \longrightarrow |\bar{X}| = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 0.3326m$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = 3.814^\circ$$

• Total solution:

• $X(t) = X_h(t) + X_p(t)$

$$x(t) = Ae^{-\zeta\omega_n t} \cos(\omega_d t - \psi) + |\bar{X}| \cos(\omega t - 0.066)$$

$$x(t) = Ae^{-0.05 \cdot 20t} \cos(19.97t - \psi) + 0.3326 \cos(20t - 0.066)$$

$$\text{at } t = 0, \quad x(t) = 0.01m$$

$$0 = A \cos \psi + 0.3326 \cos 0.066$$

$$\therefore A \cos \psi = -0.33187 \quad (1)$$

$$x(t) = A e^{-t} \cos(19.97t - \psi) + 0.3326 \cos(20t - 0.066)$$

$$\begin{aligned} \dot{x}(t) &= A e^{-t} * -19.97 \sin(19.97t - \psi) + A \cos(19.97t - \psi) * (-t) e^{-t} \\ &\quad - 0.3326 * 20 \sin(20t - 0.066) \end{aligned}$$

$$\text{at } t = 0, \quad \dot{x}(t) = 0$$

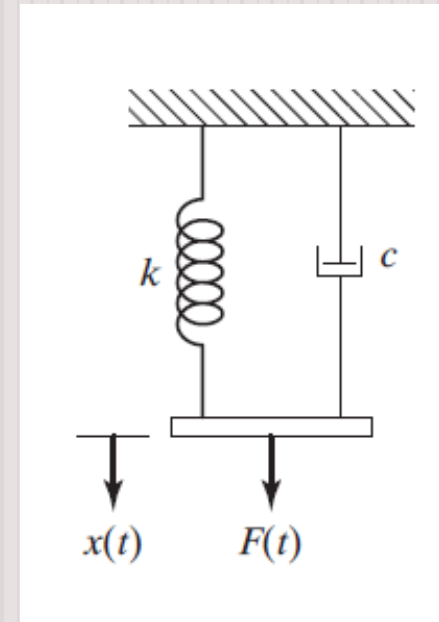
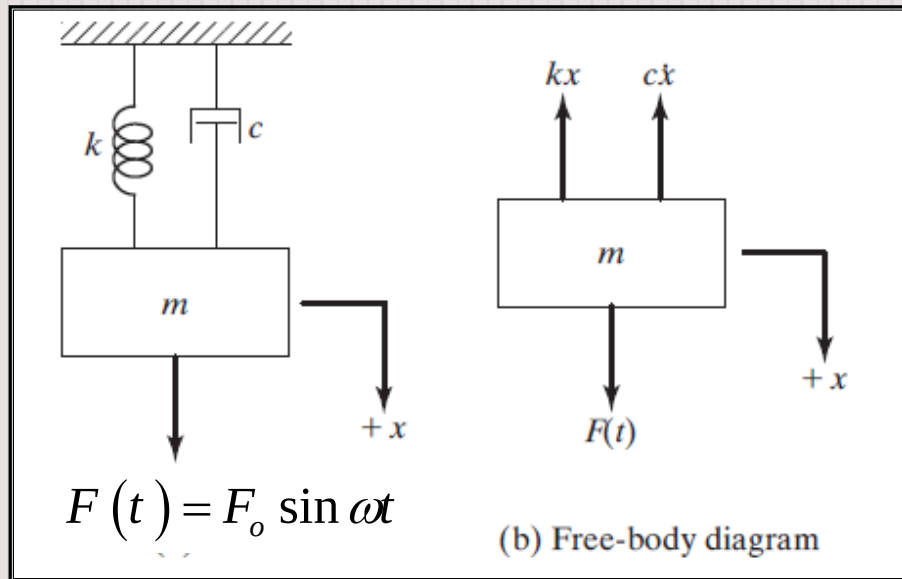
$$\therefore 0 = 19.97A \sin \psi + 0.438, \quad A \sin \psi = -0.0219 \quad (2)$$

$$\text{From (1) and (2)} \quad \psi = 0.066 \text{ rad} \quad \text{and } A = -0.3325$$

$$\therefore x(t) = -0.3325 e^{-t} \cos(19.97t - 0.066) + 0.3326 \cos(20t - 0.066)$$

ENERGY DISSIPATED BY DAMPING

Viscous Damping



$$F = kx + c\dot{x}$$

ENERGY DISSIPATED BY DAMPING

Viscous Damping

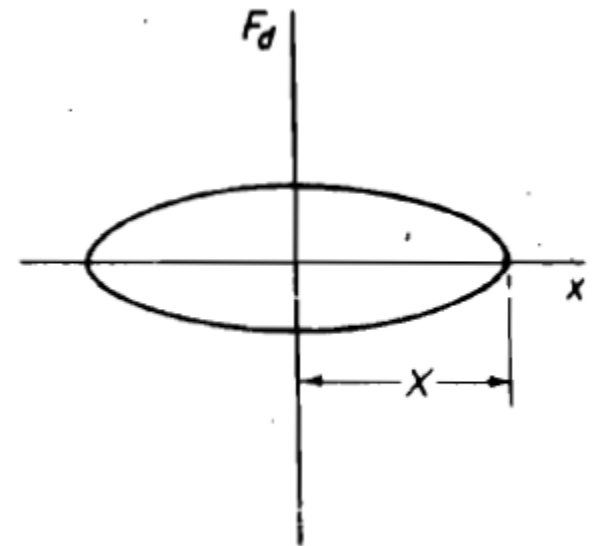
$$x = X \sin(\omega t - \phi)$$

$$\dot{x} = \omega X \cos(\omega t - \phi) = \pm \omega X \sqrt{1 - \sin^2(\omega t - \phi)}$$

$$= \pm \omega \sqrt{X^2 - x^2}$$

$$F_d = c\dot{x} = \pm c\omega \sqrt{X^2 - x^2}$$

$$\left(\frac{F_d}{c\omega X}\right)^2 + \left(\frac{x}{X}\right)^2 = 1$$



ENERGY DISSIPATED BY DAMPING

Viscous Damping

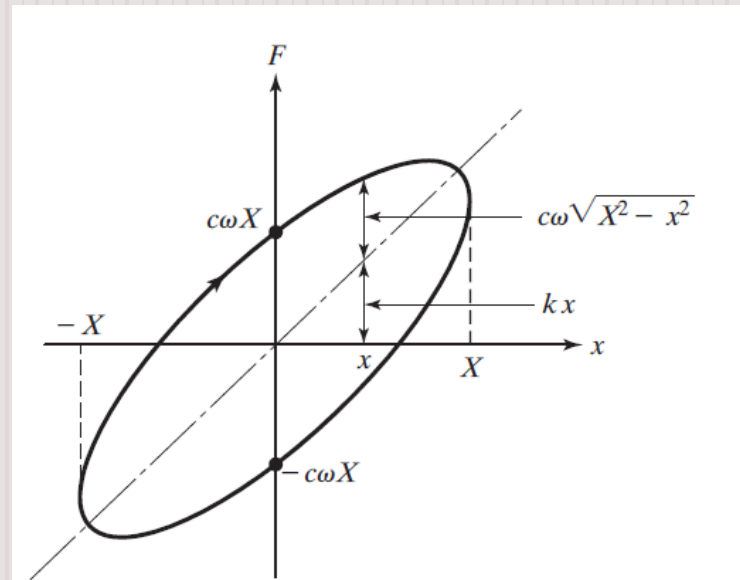
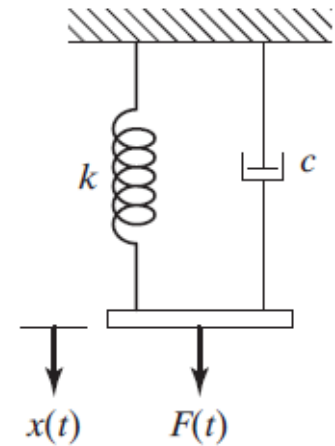
$$F = kx + c\dot{x}$$

$$x = X \sin(\omega t - \phi)$$

$$F = kX \sin(\omega t - \phi) + cX \omega \cos(\omega t - \phi)$$

$$= kx \pm c\omega \sqrt{X^2 - (X \sin(\omega t - \phi))^2}$$

$$= kx \pm c\omega \sqrt{X^2 - x^2}$$

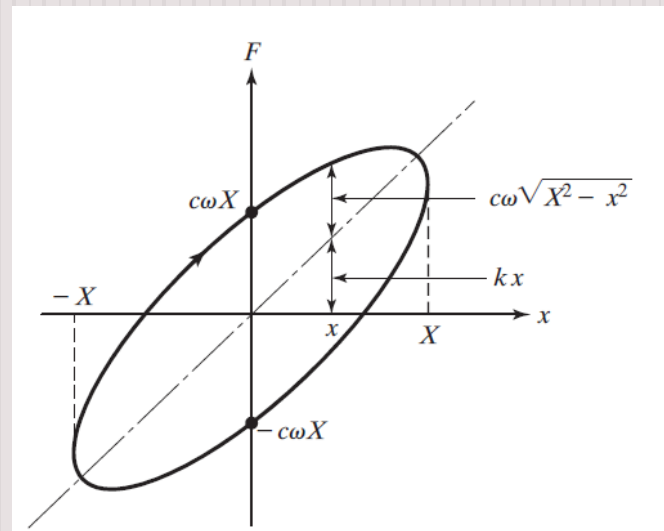


ENERGY DISSIPATED BY DAMPING

Viscous Damping

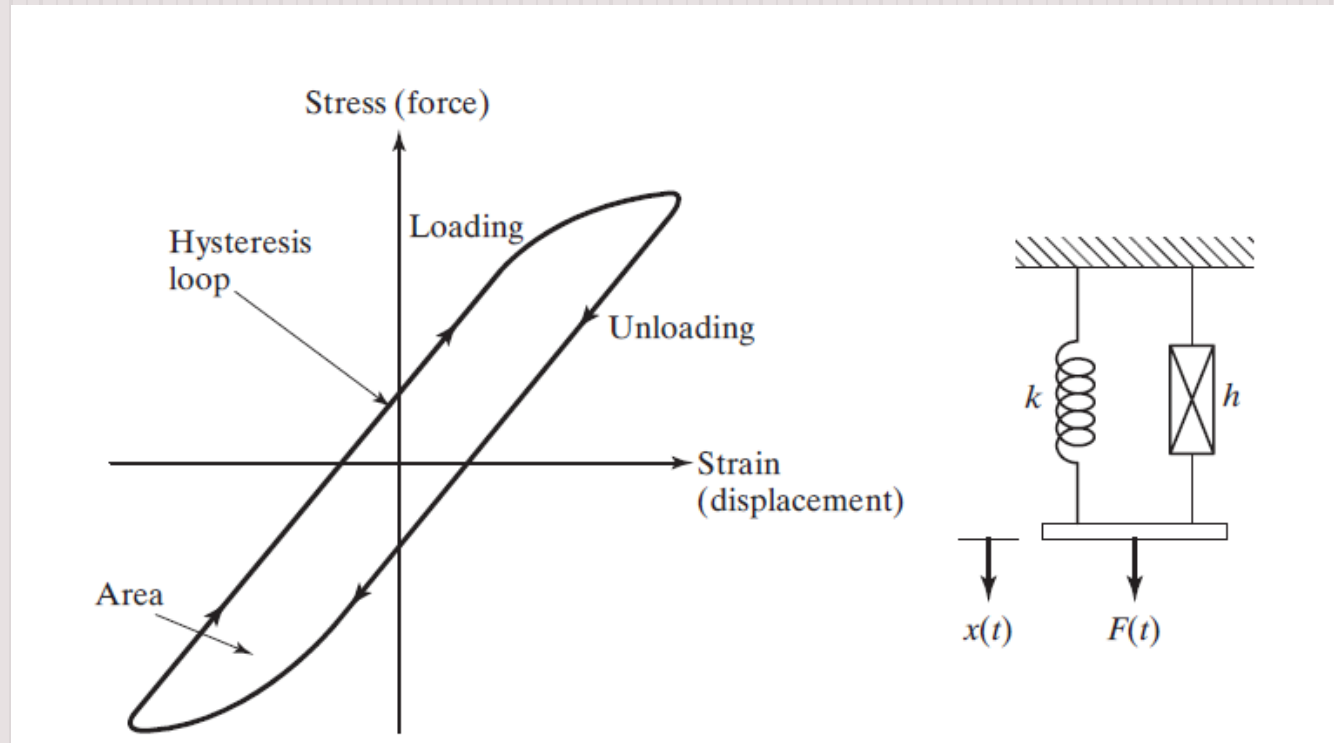
$$W_d = \oint c \dot{x} dx = \oint c \dot{x}^2 dt$$

$$= c \omega^2 X^2 \int_0^{2\pi/\omega} \cos^2(\omega t - \phi) dt = \pi c \omega X^2$$



ENERGY DISSIPATED BY DAMPING

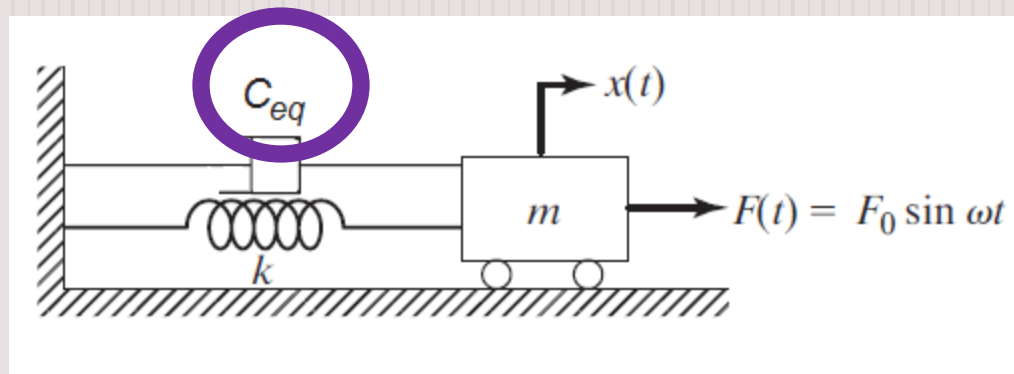
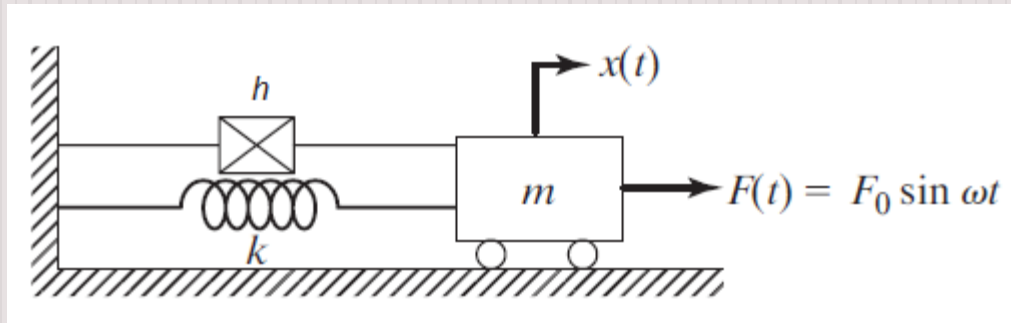
hysteresis damping



$$\Delta W = \pi h X^2$$

EQUIVALENT VISCOUS DAMPING

hysteresis damping



EQUIVALENT VISCOUS DAMPING

$$W_{d-v} = W_{d-s}$$

$$\pi C_{eq} \omega X^2 = \pi h X^2$$



$$C_{eq} = \frac{h}{\omega}$$

$$F = kx + c\dot{x}$$

$$x = Xe^{i\omega t}$$

$$F = kXe^{i\omega t} + c\omega iXe^{i\omega t} = (k + i\omega c)x$$

$$F = (k + ih)x$$

$$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta)$$

$$\rightarrow h = \beta k$$

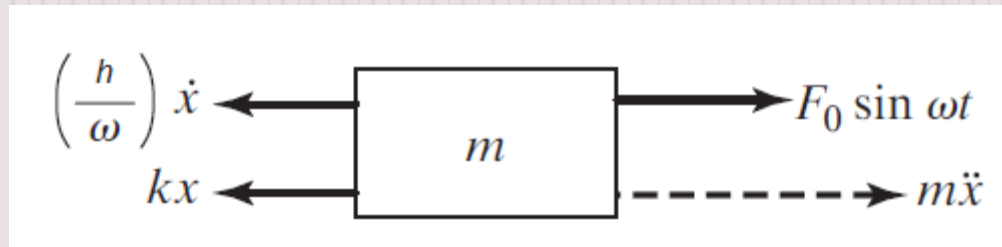
EQUIVALENT VISCOUS DAMPING

$$W_{d-v} = W_{d-s}$$

$$\pi C_{eq} \omega X^2 = \pi h X^2$$



$$C_{eq} = \frac{h}{\omega}$$



$$-\left(\frac{h}{\omega}\right)\dot{x} - kx + F_0 \sin(\omega t) = m\ddot{x}$$



$$m\ddot{x} + \left(\frac{h}{\omega}\right)\dot{x} + kx = F_0 \sin(\omega t)$$

EQUIVALENT VISCOUS DAMPING

$$m\ddot{x} + \left(\frac{h}{\omega}\right)\dot{x} + kx = F_0 \sin(\omega t)$$

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 \sin \omega t$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

$$x_p(t) = X \sin(\omega t - \phi)$$

$$x_p(t) = X \sin(\omega t - \phi)$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1} \left(\frac{c\omega}{k - m\omega^2} \right)$$

EQUIVALENT VISCOUS DAMPING

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 \sin \omega t$$

$$x_p(t) = X \sin(\omega t - \phi)$$

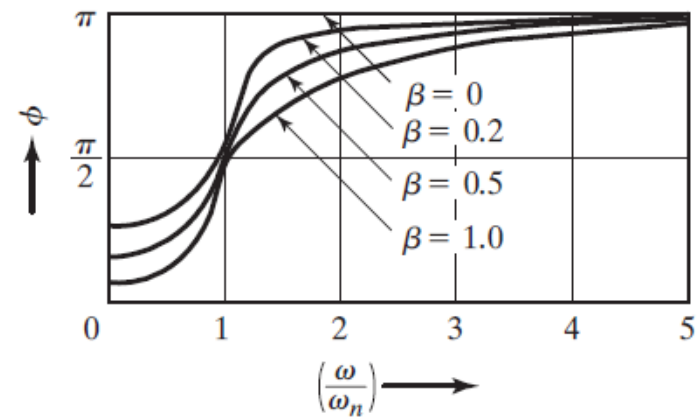
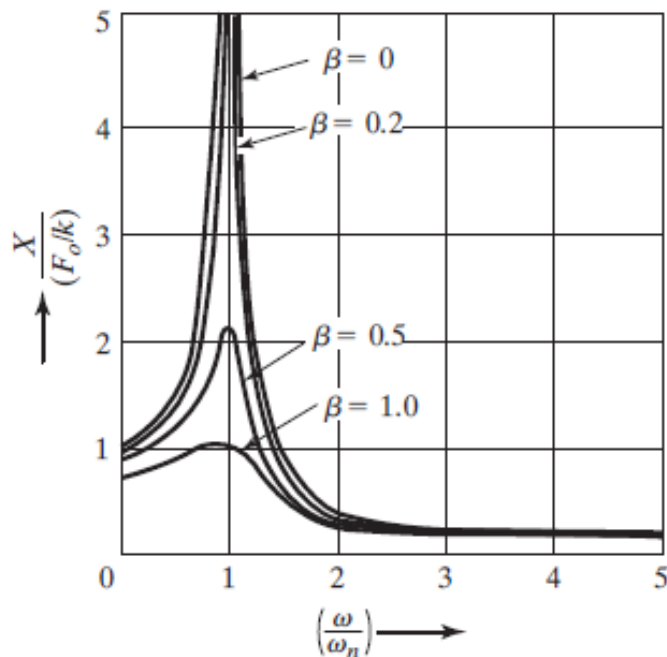
$$X = \frac{F_0}{k \left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \beta^2 \right]^{1/2}}$$

$$\phi = \tan^{-1} \left[\frac{\beta}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)} \right]$$

EQUIVALENT VISCOUS DAMPING

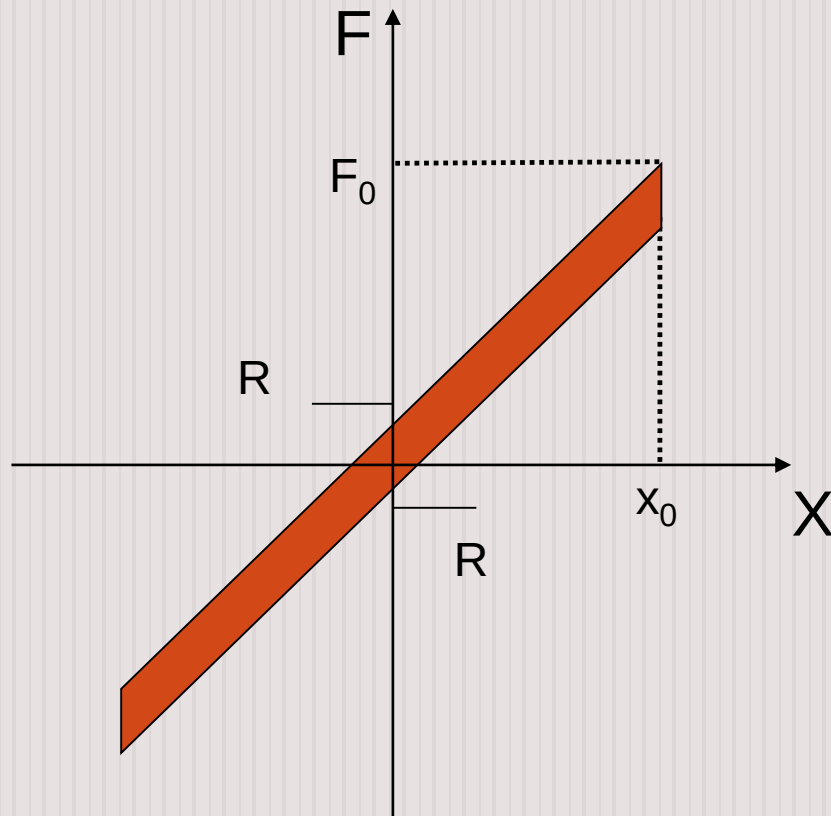
$$X = \frac{F_0}{k \left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \beta^2 \right]^{1/2}}$$

$$\phi = \tan^{-1} \left[\frac{\beta}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)} \right]$$



EQUIVALENT VISCOUS DAMPING

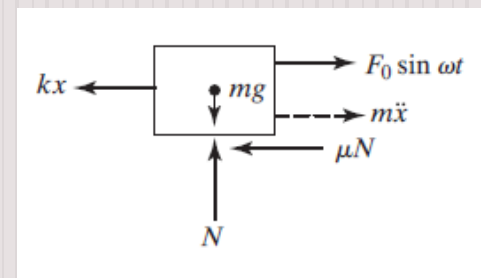
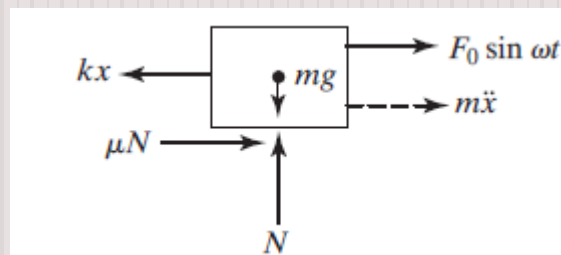
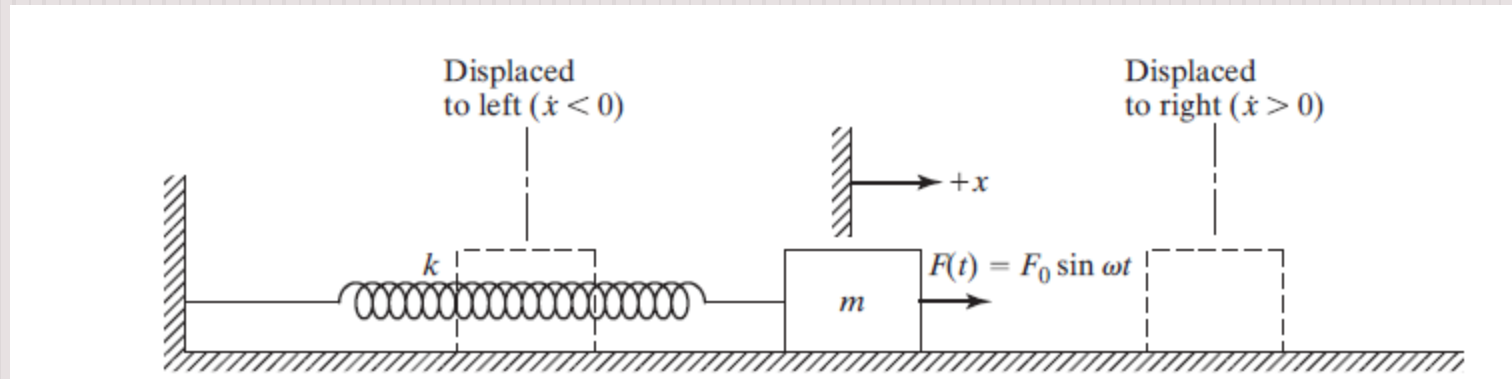
Coulomb damping



$$\Delta E = 4Rx_0$$

EQUIVALENT VISCOUS DAMPING

Coulomb damping



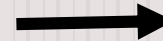
$$m\ddot{x} + kx \pm \mu N = F(t) = F_0 \sin \omega t$$

EQUIVALENT VISCOUS DAMPING

$$m\ddot{x} + kx \pm \mu N = F(t) = F_0 \sin \omega t$$

$$\Delta W = 4\mu N X$$

$$\Delta W = \pi c_{\text{eq}} \omega X^2$$



$$c_{\text{eq}} = \frac{4\mu N}{\pi \omega X}$$



$$m\ddot{x} + \left(\frac{4\mu N}{\pi \omega X} \right) \dot{x} + kx = F_0 \sin(\omega t)$$

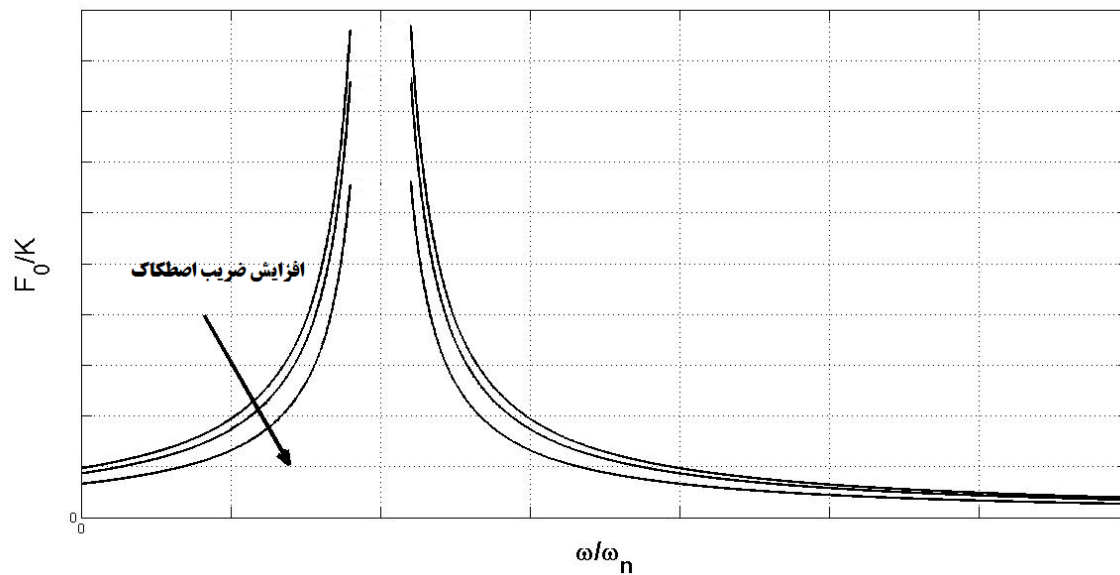
$$x_p(t) = X \sin(\omega t - \phi)$$

$$X = \frac{(F_0/k)}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(\frac{4\mu N}{\pi k X} \right)^2 \right]^{1/2}}$$

$$\phi = \tan^{-1} \left(\frac{c_{\text{eq}} \omega}{k - m\omega^2} \right) = \tan^{-1} \left\{ \frac{\frac{4\mu N}{\pi k X}}{1 - \frac{\omega^2}{\omega_n^2}} \right\}$$

EQUIVALENT VISCOUS DAMPING

$$X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4\mu N}{\pi F_0} \right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2} \right]^{1/2}$$



$$\phi = \pm \tan^{-1} \left[\frac{\frac{4\mu N}{\pi F_0}}{\left\{ 1 - \left(\frac{4\mu N}{\pi F_0} \right)^2 \right\}^{1/2}} \right]$$

